

Fluctuation limits of strongly degenerate branching systems

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Abstract

Functional limit theorems for scaled occupation time fluctuations of a sequence of generalized branching particle systems in \mathbb{R}^d with anisotropic space motions and strongly degenerated splitting abilities are studied in the cases of critical and intermediate dimensions. The results show that the limit processes are time-independent measure-valued Wiener processes with simple spatial structure.

Keywords: Functional limit theorem; Occupation time fluctuation; Branching particle system

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1. Introduction

Consider a kind of generalized branching particle systems in \mathbb{R}^d . Particles start off at time $t = 0$ from a Poisson random field with Lebesgue intensity measure λ and evolve independently. They move in \mathbb{R}^d according to a Lévy process

$$\vec{\xi} = \{\vec{\xi}(t), t \geq 0\} = \{(\xi_1(t), \xi_2(t), \dots, \xi_d(t)), t \geq 0\}$$

with independent stable components as in [17], i.e. for every $0 < k \leq d$, $\xi_k = \{\xi_k(t), t \geq 0\}$ being a symmetric α_k -stable Lévy process and ξ_1, \dots, ξ_d independent of each other. In addition, the particles split at a rate γ and the branching law at age t has the generating function

$$g(s, t) = \left(1 - \frac{e^{-\delta t}}{2}\right) + e^{-\delta t} \frac{s^2}{2}, \quad 0 \leq s \leq 1, t \geq 0.$$

Intuitively, in this model, the particles' movement in different direction is controlled by different mechanism and their probability of splitting new particles declines with the rate δ as their ages increase. It is easy to see that when $\delta = 0$, this model is similar to a classical (d, α, β) -branching particle system with $\beta = 1$ except that the moving mechanism is the anisotropic stable Lévy processes $\vec{\xi}$ rather than a symmetric α -stable Lévy process. Li and Xiao [14] called this model as a $(d, \vec{\alpha}, \delta, \gamma)$ -degenerate branching particle system, where $\vec{\alpha} := (\alpha_1, \dots, \alpha_d)$. Let $\bar{\alpha} := \sum_{k=1}^d 1/\alpha_k$. When $\bar{\alpha} > 2$, $\bar{\alpha} = 2$ and $\bar{\alpha} \in (1, 2)$, the corresponding dimension of the space is referred to as the large dimension, critical dimension and intermediate dimension, respectively.

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Motivated by the work on occupation time fluctuations of classical branching particle systems and the work on construction of anisotropic random fields (see, for example, [1]), Li and Xiao [14] explicitly studied the functional limits of occupation time fluctuations of the models. Observe that a fixed $(d, \vec{\alpha}, \delta, \gamma)$ -branching particle system with $\delta > 0$ will go to local extinction as time elapses because of the sub-critical branching laws at positive ages. They [14] borrowed the idea of nearly critical branching processes (see [12, 13, 18]) and considered a sequence of $(d, \vec{\alpha}, \delta_n, \gamma)$ -models with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. More precisely, let $N_n(s)$ denote the empirical measure of the $(d, \vec{\alpha}, \delta_n, \gamma)$ -degenerate branching particle system at time s , i.e. $N_n(s)(A)$ is the number of particles in the set $A \subset \mathbb{R}^d$ at time s . They studied the limit of a sequence of scaled occupation time fluctuations,

$$X_n(t) = \frac{1}{F_n} \int_0^{nt} (N_n(s) - f_n(s)\lambda) ds, \quad (1.1)$$

where F_n is a scaling constant and

$$f_n(s) := \bar{f}_n(s)e^{-\delta_n s} := \left[1 + \frac{\delta_n}{\gamma - \delta_n} (1 - e^{-(\gamma - \delta_n)s}) \right] e^{-\delta_n s}, \quad (1.2)$$

under the assumption $n\delta_n \rightarrow \theta \in [0, \infty)$ which is referred to weak degeneration, and proved that in the cases of critical and intermediate dimensions the limit processes have complicated temporal structures and in the case of large dimensions, the limit processes own simple temporal but anisotropic spatial structures.

The purpose of this paper is to continue the discussion of functional limits of (1.1) under the assumption that $n^\kappa \delta_n \rightarrow \theta \in (0, \infty)$ for some $\kappa \in (0, 1)$, which is referred to strong degeneration. We focus on the cases of critical and intermediate dimensions in this paper. The main methods used in this situation is same as that in Li and Xiao [14], which was formulated and developed by Bojdecki *et al* in their serial papers ([2]-[5]), except some complexities and differences from the strong degeneration. We find that the limit processes in any positive time interval are time-independent measure-valued Wiener processes, which always have the form $C\lambda\xi$, where ξ is a standard normal random variable, λ is the Lebesgue measure in \mathbb{R}^d and C is a non-random constant. By comparison with the corresponding results in Li and Xiao [14], the current limit processes are simpler (please see Remark 2.1 in Section 2 for more details). To save the space of this paper, we leave the study on the case of large dimensions elsewhere because the potential limit processes deserve further investigations. In addition, we remark that there are few results under the assumption that $\delta_n \rightarrow 0$ and $n^\kappa \delta_n \rightarrow \infty$ for any $\kappa > 0$.

There is much literature related to the field of fluctuations of branching particle systems. Iscoe [11] studied the single time limit theorem of occupation time of the (d, α, β) -superprocess which in essence is the limit process of the classical (d, α, β) -branching system, and got different limits depending on the relations between d, α, β . Hong [16] also considered the superprocess case and proved the convergence of finite-dimensional distributions of real processes without the tightness for a fixed test function. Recently Bojdecki *et al* in their series of papers, such as [2]-[7], studied the functional limits of occupation time fluctuations of a fixed classical (d, α, β) -branching system which is different from the setting in Li and Xiao [14] and this paper. For more literature we refer to [8, 9, 15] and the references therein.

Without other statement, in this paper, we use K to denote an unspecified positive finite constant which may not necessarily be the same in each occurrence. In addition, since this

paper and [14] discuss the same branching systems in different assumptions, in order to shorten the length of this paper, we will omit some common inferences and calculations and refer to [14].

The remainder of this paper is organized as follows. Section 2 contains the main results of this paper and some auxiliary results and formulas used in the proofs of the main results. In Section 3 we prove the main results.

2. Main results

Consider a sequence of $(d, \vec{\alpha}, \delta_n, \gamma)$ -degenerate branching particle system. The particles' spatial movement is described by $\vec{\xi}_n$. We assume that $\{\vec{\xi}_n, n \geq 1\}$ is a sequence of identically distributed \mathbb{R}^d -valued Lévy processes with α_k -stable components ($1 \leq k \leq d$). The distribution of $\vec{\xi}_n$ is completely determined by its characteristic function

$$\mathbb{E}\left(e^{i\langle z, \vec{\xi}_n(t) \rangle}\right) = e^{-t \sum_{k=1}^d |z_k|^{\alpha_k}}, \quad z \in \mathbb{R}^d. \quad (2.1)$$

Obviously, for any $n > 0$, $\vec{\xi}_n$ is a time-homogeneous Markov process on \mathbb{R}^d . Since $\vec{\xi}_n$ has the same distribution for all n , we denote its semigroup by $\{T_t\}_{t \geq 0}$, i.e.,

$$T_s f(x) := \mathbb{E}(f(\vec{\xi}_n(t+s)) | \vec{\xi}_n(t) = x),$$

for all $s, t \geq 0$, $x \in \mathbb{R}^d$ and bounded measurable functions f on \mathbb{R}^d . In order to avoid misunderstanding, in case of necessity we write $T_s f(x)$ by $T_s(f(\cdot))(x)$.

In this paper, we always let $N_n(t)$ be the empirical measure of the $(d, \vec{\alpha}, \delta_n, \gamma)$ -model for every $n \geq 1$. From Section 2 in Li and Xiao [14], we know that the scaled occupation time fluctuations of $(d, \vec{\alpha}, \delta_n, \gamma)$ -models are defined as follows.

$$\langle X_n(t), \phi \rangle = \frac{1}{F_n} \int_0^{nt} \langle N_n(s) - f_n(s)\lambda, \phi \rangle ds, \quad (2.2)$$

for every $\phi \in \mathcal{S}(\mathbb{R}^d)$, the space of smooth rapidly decreasing functions, where F_n is a suitable scaling parameter, $f_n(s)$ same as (1.2) and $\langle \mu, f \rangle = \int f d\mu$ for any measure μ and any integrable function f on μ . Furthermore,

$$\mathbb{E}(\langle N_n(s), \phi \rangle | N_0 = \epsilon_x) = f_n(s) T_s \phi(x), \quad (2.3)$$

for all $n \geq 0, x \in \mathbb{R}^d$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Here ϵ_x denotes the unit measure concentrated at $x \in \mathbb{R}^d$.

Below, we assume that there is a constant $\theta \in (0, \infty)$ such that $n^\kappa \delta_n \rightarrow \theta$ for some $k \in (0, 1)$ as $n \rightarrow \infty$. Let $\widehat{\phi}(z)$ ($z \in \mathbb{R}^d$) be the Fourier transform of function $\phi \in L(\mathbb{R}^d)$, i.e., $\widehat{\phi}(z) = \int_{\mathbb{R}^d} e^{i\langle x, z \rangle} \phi(x) dx$, and $\mathcal{S}'(\mathbb{R}^d)$ the dual space of $\mathcal{S}(\mathbb{R}^d)$. Recall that $\vec{\alpha} := (\alpha_1, \dots, \alpha_d)$ and $\bar{\alpha} := \sum_{k=1}^d 1/\alpha_k$. The main results of this paper read as follows.

Theorem 2.1 *When $\bar{\alpha} = 2$, let $F_n^2 = n^\kappa \ln n$. Then for any $\varepsilon > 0$, $X_n \Rightarrow C_1 \lambda \zeta$ in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^d))$ as $n \rightarrow \infty$, where ζ is a standard normal random variable and*

$$C_1 = \sqrt{\frac{2\gamma\kappa}{\theta(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{(1 + \sum_{k=1}^d |y_k|^{\alpha_k})^3} dy}.$$

Theorem 2.2 When $1 < \bar{\alpha} < 2$, let $F_n^2 = n^{(3-\bar{\alpha})\kappa}$. Then for any $\varepsilon > 0$, $X_n \Rightarrow C_2 \lambda \zeta$ in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^d))$ as $n \rightarrow \infty$, where ζ is a standard normal random variable and

$$C_2 = \sqrt{\frac{\gamma}{\pi^d} \prod_{k=1}^d \frac{\Gamma(1/\alpha_k)}{\alpha_k} \int_0^\infty e^{-\theta u} du \int_0^\infty e^{-\theta v} dv \int_0^{v \wedge u} \frac{e^{\theta s} ds}{(u+v-2s)^{\bar{\alpha}}}.$$

Remark 2.1 (1) Compared with the corresponding results in the case of weak degeneration (see [14, Theorem 2.1, Theorem 2.2 and Remark 2.1]), the limit processes are simpler in the temporal structure.

(2) Though Li and Xiao [14] pointed out that their results under the case of $\bar{\alpha} = 2$ can be strengthened to the weak functional convergence in $C([0, 1], \mathcal{S}'(\mathbb{R}^d))$ by a lengthy and tedious method, due to the strong degeneration, in this paper, we can use a relatively simple way to get the weak convergence in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^d))$.

(3) Note that $X_n(\cdot) \in C([0, 1], \mathcal{S}'(\mathbb{R}^d))$ and $X_n(0) = 0$ and that X is a non-zero time-independent measure-valued Wiener processes. X_n does not weakly converge to X in $C([0, 1], \mathcal{S}'(\mathbb{R}^d))$ because the limit of X_n in $C([0, 1], \mathcal{S}'(\mathbb{R}^d))$ must be continuous and its initial value has to be 0 a.s.

(4) For the case of large dimensions a similar result holds, i.e., if $\bar{\alpha} > 2$ and $F_n^2 = n^\kappa$, then, for any $\varepsilon > 0$, $X_n \Rightarrow X$ in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^d))$ as $n \rightarrow \infty$, where X is a centered time-independent Gaussian process valued in $\mathcal{S}'(\mathbb{R}^d)$, with covariance function

$$\begin{aligned} & \text{Cov}(\langle X(s), \phi_1 \rangle, \langle X(t), \phi_2 \rangle) \\ &= \frac{1}{\theta(2\pi)^d} \int_{\mathbb{R}^d} \left[\frac{2}{\sum_{k=1}^d |z_k|^{\alpha_k}} + \frac{\gamma}{(\sum_{k=1}^d |z_k|^{\alpha_k})^2} \right] \widehat{\phi}_1(z) \overline{\widehat{\phi}_2(z)} dz. \end{aligned}$$

It is also interesting to study the properties of the limit processes. We will discuss these problems elsewhere.

For the convenience of reference, at the end of this section, we collect some formulas and results as follows.

Lemma 2.1 ([14, Remark 2.3]) Let $z = (z_1, \dots, z_d)$. For any $\{\alpha_k > 0, k = 1, \dots, d\}$, if $0 < r < \bar{\alpha}$, then $\int_{[0,1]^d} \frac{1}{\sum_{k=1}^d |z_k|^{r\alpha_k}} dz < \infty$, and if $r > \bar{\alpha}$, then $\int_{\mathbb{R}^d \setminus [0,1]^d} \frac{1}{\sum_{k=1}^d |z_k|^{r\alpha_k}} dz < \infty$. Therefore, if $\tau(z)$ is bounded and $\int_{\mathbb{R}^d} \tau(z) dz < \infty$, then $\int_{\mathbb{R}^d} \frac{\tau(z)}{\sum_{k=1}^d |z_k|^{r\alpha_k}} dz < \infty$, for all $r \in (0, \bar{\alpha})$.

Let ϕ_1, ϕ_2 and ϕ_3 be functions from \mathbb{R}^d to \mathbb{R} , bounded and integrable. Then

$$\int_{\mathbb{R}^d} \phi_1(x) \phi_2(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\phi}_1(z) \overline{\widehat{\phi}_2(z)} dz, \quad (2.4)$$

(the Plancherel formula). Furthermore, if $\widehat{\phi}_1$ and $\widehat{\phi}_2$ are integrable, then

$$\int_{\mathbb{R}^d} \phi_1(x) \phi_2(x) \phi_3(x) dx = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \widehat{\phi}_1(z) \widehat{\phi}_2(z_1) \overline{\widehat{\phi}_3(z+z_1)} dz dz_1, \quad (2.5)$$

(the inverse Fourier transform), and that, by the Riemann-Lebesgue Lemma, $\widehat{\phi}_1(z)$ is bounded and goes to 0 as $|z| \rightarrow \infty$.

Since components of $\vec{\xi}$ are symmetric stable Lévy processes and independent of each other, for any $t > 0$

$$\int_{\mathbb{R}^d} \phi_1(x) T_t \phi_2(x) dx = \int_{\mathbb{R}^d} \phi_2(x) T_t \phi_1(x) dx, \quad (2.6)$$

and

$$\widehat{T_t \phi_1}(z) = \widehat{\phi_1}(z) e^{-t \sum_{k=1}^d |z_k|^{\alpha_k}}. \quad (2.7)$$

3. The proofs of main results

First of all, we define a sequence of random variables \tilde{X}_n in $\mathcal{S}'(\mathbb{R}^{d+1})$ as follows:

For any $n \geq 0$ and $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$, let

$$\langle \tilde{X}_n, \psi \rangle = \int_0^1 \langle X_n(t), \psi(\cdot, t) \rangle dt. \quad (3.1)$$

In order to prove the main results, as what Bojdecki et al did in their serial papers ([2]-[5]), we need show the following facts.

(i) $\langle \tilde{X}_n, \psi \rangle$ converges in distribution to $\langle \tilde{X}, \psi \rangle$ for all $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$ as $n \rightarrow \infty$, where \tilde{X}_n and \tilde{X} are defined as (3.1) and X is the corresponding limit process.

(ii) For any given $\varepsilon > 0$, $\{\langle X_n, \phi \rangle, n \geq 1\}$ is tight in $C([\varepsilon, 1])$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where the theorem of Mitoma [16] is used.

As explained in Bojdecki et al [2], (i) will be proved if we show that

$$\lim_{n \rightarrow \infty} \mathbb{E}(e^{-\langle \tilde{X}_n, \psi \rangle}) = \exp \left\{ \frac{1}{2} \int_0^1 \int_0^1 Cov(\langle X(s), \psi(\cdot, s) \rangle, \langle X(t), \psi(\cdot, t) \rangle) ds dt \right\}, \quad (3.2)$$

for each non-negative $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$.

Below, we state the proof of Theorem 2.1 in detail. Since the proof of Theorem 2.2 is similar and easier, we omit it.

Proof of Theorem 2.1. To prove (3.2), we assume $\psi(x, t) = \phi(x)h(t)$, where $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $h \in \mathcal{S}(\mathbb{R})$ are arbitrary given nonnegative functions. For general ψ , the proof is the same with slightly more complicated notation.

Now, we recall some formulas from Li and Xiao [14] as follows.

$$\begin{aligned} \mathbb{E}(e^{-\langle \tilde{X}_n, \psi \rangle}) &= \exp \left\{ \int_{\mathbb{R}^d} dx \int_0^n f_n(s) T_s \psi_n(x, s) ds - \int_{\mathbb{R}^d} [1 - H_{n, \psi_n}(x, n, 0)] dx \right\} \\ &= \exp \left\{ \int_{\mathbb{R}^d} [J_{n, \psi_n}(x, n, 0) - V_{n, \psi_n}(x, n, 0)] dx \right\} \\ &= \exp \{ I_1(n, \psi_n) + I_2(n, \psi_n) + I_3(n, \psi_n) \}, \end{aligned} \quad (3.3)$$

where

$$I_1(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} V_{n, \psi_n}^2(x, n-s, s) ds, \quad (3.4)$$

$$I_2(n, \psi_n) = \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} \psi_n(x, s) V_{n, \psi_n}(x, n-s, s) ds, \quad (3.5)$$

$$I_3(n, \psi_n) = \delta_n \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} \chi_{n, \psi_n}(x, u, s) du. \quad (3.6)$$

Here

$$\psi_n(x, s) = \frac{1}{F_n} \phi(x) \tilde{h}\left(\frac{s}{n}\right) \quad \text{and} \quad \tilde{h}(s) = \int_s^1 h(t) dt, \quad (3.7)$$

$$V_{n, \psi_n}(x, t, r) = 1 - \mathbb{E}_x \left(\exp \left\{ - \int_0^t \langle N_n(s), \psi_n(\cdot, r+s) \rangle ds \right\} \right), \quad (3.8)$$

and

$$\chi_{n, \psi_n}(x, u, s) = \mathbb{E}_x \left[\left(1 - e^{-\int_0^u \psi_n(\vec{\xi}_n(v), s+v) dv} \right) \psi_n(\vec{\xi}_n(u), s+u) \right]. \quad (3.9)$$

In addition, for any $x \in \mathbb{R}^d$ and $t, s \geq 0$, from Li and Xiao [14] we still have that

$$V_{n, \psi_n}(x, t, r) \leq \int_0^t f_n(s) T_s \psi_n(\cdot, r+s)(x) ds =: J_{n, \psi_n}(x, t, r), \quad (3.10)$$

and that

$$\begin{aligned} J_{n, \psi_n}(x, t, r) - V_{n, \psi_n}(x, t, r) &= \delta_n \int_0^t e^{-\delta_n s} T_s \left(\int_0^{t-s} e^{-\gamma u} \chi_n(\cdot, u, r+s) du \right) (x) ds \\ &\quad + \int_0^t e^{-\delta_n s} T_s \left\{ \psi_n(\cdot, r+s) V_{n, \psi_n}(\cdot, t-s, r+s) \right\} (x) ds \\ &\quad + \frac{\gamma}{2} \int_0^t e^{-\delta_n s} T_s V_{n, \psi_n}^2(x, t-s, r+s) ds. \end{aligned} \quad (3.11)$$

Below, we discuss the limits of $I_1(n, \psi_n)$, $I_2(n, \psi_n)$ and $I_3(n, \psi_n)$, respectively. We remind that for all $t > 0$ and $y = t^H z$, where H be the $d \times d$ diagonal matrix $(1/\alpha_k)_{1 \leq k \leq d}$,

$$dy = t^2 dz.$$

Step 1 We are going to get the limit of $I_1(n, \psi_n)$. From (3.4), we get that

$$I_1(n, \psi_n) = I_{11}(n, \psi_n) + I_{12}(n, \psi_n), \quad (3.12)$$

where

$$I_{11}(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} J_{n, \psi_n}^2(x, n-s, s) ds, \quad (3.13)$$

$$I_{12}(n, \psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} (V_{n, \psi_n}^2(x, n-s, s) - J_{n, \psi_n}^2(x, n-s, s)) ds. \quad (3.14)$$

We first consider the limit of $I_{11}(n, \psi_n)$. Substituting (1.2), (3.7) and (3.10) into (3.13), we get that

$$I_{11}(n, \psi_n) = \frac{n^3 \gamma 2}{2 F_n^2} \int_0^1 e^{-n \delta_n s} ds \int_{\mathbb{R}^d} \left[\int_0^{1-s} \bar{f}_n(nu) e^{-n \delta_n u} \tilde{h}(s+u) T_{nu} \phi(x) du \right]^2 dx.$$

Furthermore applying (2.4), (2.6) and (2.7) to the above formula, and noting that $\bar{f}_n(u)$ converges uniformly to 1 as $n \rightarrow \infty$, we derive that

$$\begin{aligned} \lim_{n \rightarrow \infty} I_{11}(n, \psi_n) &= \lim_{n \rightarrow \infty} \frac{n^3 \gamma}{2 F_n^2} \int_0^1 e^{-n \delta_n s} ds \int_{\mathbb{R}^d} \left[\int_0^{1-s} e^{-n \delta_n u} \tilde{h}(s+u) T_{nu} \phi(x) du \right]^2 dx \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{n^3 \gamma / 2}{(2\pi)^d F_n^2} \int_0^1 e^{-n \delta_n s} ds \int_{\mathbb{R}^d} |\widehat{\phi}(z)|^2 \right. \\ &\quad \times \left[\int_s^1 e^{-n(u-s)(\delta_n + \sum_{k=1}^d |z_k|^{\alpha_k})} \tilde{h}(u) du \right]^2 dz \Big\}. \end{aligned} \quad (3.15)$$

Substituting $\tilde{h}(u) = \int_u^1 h(t)dt$ and $F_n^2 = n^\kappa \ln n$ into (3.15), by changing the integral order we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} I_{11}(n, \psi_n) &= \lim_{n \rightarrow \infty} \left\{ \frac{n\gamma/2}{(2\pi)^d F_n^2} \int_0^1 h(r)dr \int_0^1 h(t)dt \int_0^{t \wedge r} e^{-n\delta_n s} ds \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 \right. \\ &\quad \times \frac{(1 - e^{-n(r-s)(\delta_n + \sum_{k=1}^d |z_k|^{\alpha_k})})(1 - e^{-n(t-s)(\delta_n + \sum_{k=1}^d |z_k|^{\alpha_k})})}{(\delta_n + \sum_{k=1}^d |z_k|^{\alpha_k})^2} dz \Big\} \\ &= \lim_{n \rightarrow \infty} \frac{\gamma}{(2\pi)^d} \int_0^1 h(r)dr \int_0^r h(t)dt \int_0^\infty e^{-n^\kappa \delta_n s} \frac{W_{n,r,t,s}(n^\kappa \delta_n)}{\ln n} ds, \end{aligned} \quad (3.16)$$

where for any $x > 0$, $0 \leq t \leq r \leq 1$ and $s > 0$,

$$W_{n,r,t,s}(x) = \begin{cases} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 \Phi(x, s, n^{1-\kappa}r, n^{1-\kappa}t, n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k}) n^{2\kappa} dz, & s < n^{1-\kappa}t; \\ 0, & s \geq n^{1-\kappa}t, \end{cases}$$

and for any $x > 0$, $s, u, v, y \in [0, \infty)$

$$\Phi(x, s, u, v, y) = \frac{(1 - e^{-(u-s)(x+y)})(1 - e^{-(v-s)(x+y)})}{(x+y)^2}.$$

Let

$$\tilde{W}(n) = \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 \frac{(1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}})^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^2 \ln n} dz. \quad (3.17)$$

It is easy to see that

$$W_{n,r,t,s}(n^\kappa \delta_n) / \ln n \leq \tilde{W}(n)$$

for all $(r, t, s) \in \{0 \leq t \leq r \leq 1; 0 \leq s\}$, where we use the decreasing of $(1 - e^{-r})/r$ on $r \in (0, +\infty)$. Furthermore, applying L'Hôpital's law, we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{W}(n) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} 2n |\hat{\phi}(z)|^2 \frac{(1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}}) e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}}}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz \\ &= 2 |\hat{\phi}(0)|^2 \int_{\mathbb{R}^d} \frac{e^{-\sum_{k=1}^d |z_k|^{\alpha_k}} (1 - e^{-\sum_{k=1}^d |z_k|^{\alpha_k}})}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz < \infty, \end{aligned} \quad (3.18)$$

and hence $\{\tilde{W}(n)\}$ is bounded. Therefore, the dominated convergence theorem plus the convergence of $n^\kappa \delta_n \rightarrow \theta$ yields that if

$$W_{n,r,t,s}(n^\kappa \delta_n) / \ln n \rightarrow 2\kappa \int_{\mathbb{R}^d} \frac{|\hat{\phi}(0)|^2}{(1 + \sum_{k=1}^d |y_k|^{\alpha_k})^3} dy, \quad a.s. \quad (3.19)$$

on $(r, t, s) \in \{0 \leq t \leq r \leq 1, 0 \leq s\}$, then

$$\begin{aligned} I_{11}(n, \psi_n) &\rightarrow \frac{2\gamma\kappa}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(0)|^2}{(1 + \sum_{k=1}^d |y_k|^{\alpha_k})^3} dy \int_0^1 h(r)dr \int_0^r h(t)dt \int_0^\infty e^{-\theta s} ds \\ &= \frac{\gamma\kappa}{\theta(2\pi)^d} \int_{\mathbb{R}^d} \frac{dy}{(1 + \sum_{k=1}^d |y_k|^{\alpha_k})^3} \left(\int_{\mathbb{R}^d} \phi(x) dx \int_0^1 h(t)dt \right)^2. \end{aligned} \quad (3.20)$$

Below, we prove (3.19). To this end, by the mean-value theorem and using the substitution $y = \Theta_n z := (n^\kappa)^H z$ we have that

$$\begin{aligned} &|W_{n,r,t,s}(x_1) - W_{n,r,t,s}(x_2)| \\ &= \begin{cases} |x_1 - x_2| \int_{\mathbb{R}^d} |\hat{\phi}(\Theta_n^{-1}y)|^2 \Phi'_x(\vartheta, s, n^{1-\kappa}r, n^{1-\kappa}t, \sum_{k=1}^d |y_k|^{\alpha_k}) dy, & s < n^{1-\kappa}t, \\ 0, & s \geq n^{1-\kappa}t, \end{cases} \end{aligned}$$

for any $x_1, x_2 > 0$, where $\vartheta \in (x_1, x_2)$. Note that

$$\begin{aligned} & \Phi'_x(\vartheta, s, n^{1-\kappa}r, n^{1-\kappa}t, \sum_{k=1}^d |y_k|^{\alpha_k}) \\ &= \frac{(n^{1-\kappa}r - s)e^{-(n^{1-\kappa}r-s)(\vartheta+\sum_{k=1}^d |y_k|^{\alpha_k})}(1 - e^{-(n^{1-\kappa}t-s)(\vartheta+\sum_{k=1}^d |y_k|^{\alpha_k})})}{(\vartheta + \sum_{k=1}^d |y_k|^{\alpha_k})^2} \\ &+ \frac{(n^{1-\kappa}t - s)e^{-(n^{1-\kappa}t-s)(\vartheta+\sum_{k=1}^d |y_k|^{\alpha_k})}(1 - e^{-(n^{1-\kappa}r-s)(\vartheta+\sum_{k=1}^d |y_k|^{\alpha_k})})}{(\vartheta + \sum_{k=1}^d |y_k|^{\alpha_k})^2} \\ &- \frac{2(1 - e^{-(n^{1-\kappa}r-s)(\vartheta+\sum_{k=1}^d |y_k|^{\alpha_k})})(1 - e^{-(n^{1-\kappa}t-s)(\vartheta+\sum_{k=1}^d |y_k|^{\alpha_k})})}{(\vartheta + \sum_{k=1}^d |y_k|^{\alpha_k})^3}. \end{aligned}$$

We have that for any given $1 \geq r \geq t \geq 0$ and $s \geq 0$, there exists $N > 0$ such that for all $n > N$,

$$|W_{n,r,t,s}(x_1) - W_{n,r,t,s}(x_2)| \leq |x_1 - x_2| \int_{\mathbb{R}^d} |\widehat{\phi}(\Theta_n^{-1}y)|^2 Z_n(\vartheta, r, t, s, y) dy, \quad (3.21)$$

where

$$\begin{aligned} Z_n(\vartheta, r, t, s, y) &= \frac{2}{(\vartheta + \sum_{k=1}^d |y_k|^{\alpha_k})^3} + \frac{(n^{1-\kappa}r - s)e^{-(n^{1-\kappa}r-s)\sum_{k=1}^d |y_k|^{\alpha_k}}}{\vartheta^2} \\ &+ \frac{(n^{1-\kappa}t - s)e^{-(n^{1-\kappa}t-s)(\sum_{k=1}^d |y_k|^{\alpha_k})}}{\vartheta^2}. \end{aligned}$$

Because for any given r, t, s and sufficiently large n , $\int_{\mathbb{R}^d} Z_n(\vartheta, r, t, s, y) dy$ equals

$$\frac{1}{\vartheta^2} \left[\frac{1}{n^{1-\kappa}r - s} + \frac{1}{n^{1-\kappa}t - s} \right] \int_{\mathbb{R}^d} e^{-\sum_{k=1}^d |y_k|^{\alpha_k}} dy + \int_{\mathbb{R}^d} \frac{1}{(\vartheta + \sum_{k=1}^d |y_k|^{\alpha_k})^3} dy,$$

which is bounded for sufficiently large n , from $n^{1-\kappa}\delta_n \rightarrow \theta \in (0, +\infty)$ and (3.21), we obtain that as $n \rightarrow \infty$,

$$|W_{n,r,t,s}(n^{1-\kappa}\delta_n) - W_{n,r,t,s}(\theta)| \rightarrow 0, \quad (3.22)$$

for any given $(r, t, s) \in \{0 \leq t \leq r \leq 1, 0 \leq s\}$. Therefore

$$\lim_{n \rightarrow \infty} \frac{W_{n,r,t,s}(n^{1-\kappa}\delta_n)}{\ln n} = \lim_{n \rightarrow \infty} \frac{W_{n,r,t,s}(\theta)}{\ln n} = \lim_{n \rightarrow \infty} n \frac{\partial W_{n,r,t,s}(\theta)}{\partial n}, \quad (3.23)$$

where we use L'Hôpital's law at the second equality. Note that for any $(r, t, s) \in \{0 \leq t \leq r \leq 1, 0 \leq s\}$ and sufficiently large n

$$n \frac{\partial W_{n,r,t,s}(\theta)}{\partial n} = n \int_{\mathbb{R}^d} |\widehat{\phi}(z)|^2 \frac{\partial \Phi(\theta, s, n^{1-\kappa}r, n^{1-\kappa}t, n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k}) n^{2\kappa}}{\partial n} dz,$$

which, by direct calculations, equals

$$\begin{aligned}
& \int_{\mathbb{R}^d} \left\{ |\widehat{\phi}(z)|^2 \left((1-\kappa)\theta n^{1-\kappa}r + (n^{1-\kappa}r - \kappa s)n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k} \right) \right. \\
& \quad \times \frac{e^{-(n^{1-\kappa}r-s)(\theta+n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})} (1 - e^{-(n^{1-\kappa}t-s)(\theta+n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})})}{(\theta + n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})^2} \left. \right\} n^{2\kappa} dz \\
& + \int_{\mathbb{R}^d} \left\{ |\widehat{\phi}(z)|^2 \left((1-\kappa)\theta n^{1-\kappa}t + (n^{1-\kappa}t - \kappa s)n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k} \right) \right. \\
& \quad \times \frac{e^{-(n^{1-\kappa}t-s)(\theta+n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})} (1 - e^{-(n^{1-\kappa}r-s)(\theta+n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})})}{(\theta + n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})^2} \left. \right\} n^{2\kappa} dz \\
& + \int_{\mathbb{R}^d} \left\{ |\widehat{\phi}(z)|^2 \frac{2\theta\kappa(1 - e^{-(n^{1-\kappa}r-s)(\theta+n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})})}{(\theta + n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})^3} \right. \\
& \quad \times (1 - e^{-(n^{1-\kappa}t-s)(\theta+n^\kappa \sum_{k=1}^d |z_k|^{\alpha_k})}) \left. \right\} n^{2\kappa} dz.
\end{aligned}$$

Now substituting $y = (n^\kappa)^H z$ into the above formula and letting $n \rightarrow \infty$, we get that for any given r, t, s , as $n \rightarrow \infty$,

$$\begin{aligned}
n \frac{\partial W_{n,r,t,s}(\theta)}{\partial n} & \rightarrow 2\kappa |\widehat{\phi}(0)|^2 \int_{\mathbb{R}^d} \frac{\theta}{(\theta + \sum_{k=1}^d |y_k|^{\alpha_k})^3} dy \\
& = 2\kappa |\widehat{\phi}(0)|^2 \int_{\mathbb{R}^d} \frac{1}{(1 + \sum_{k=1}^d |y_k|^{\alpha_k})^3} dy,
\end{aligned}$$

which is the desired formula (3.19).

To study the limit of $I_{12}(n, \psi_n)$, we first observe that from (3.9)-(3.11),

$$\begin{aligned}
& J_{n,\psi_n}^2(x, n-s, s) - V_{n,\psi_n}^2(x, n-s, s) \\
& \leq 2 \left[\delta_n \int_0^{n-s} e^{-\delta_n u} T_u \left(\int_0^{n-s-u} e^{-\gamma v} \chi_{n,\psi_n}(\cdot, v, s+u) dv \right) (x) du \right. \\
& \quad + \int_0^{n-s} e^{-\delta_n u} T_u (\psi_n(\cdot, s+u) J_{n,\psi_n}(\cdot, n-s-u, s+u)) (x) du \\
& \quad \left. + \frac{\gamma}{2} \int_0^{n-s} e^{-\delta_n u} T_u J_{n,\psi_n}^2(x, n-s-u, s+u) du \right] J_{n,\psi_n}(x, n-s, s) \\
& =: 2J_{n,\psi_n}(x, n-s, s) \left(\bar{I}_{121}(n, \psi_n) + \bar{I}_{122}(n, \psi_n) + \bar{I}_{123}(n, \psi_n) \right). \tag{3.24}
\end{aligned}$$

Let

$$I_{121}(n, \psi_n) := \gamma \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} J_{n,\psi_n}(x, n-s, s) \bar{I}_{121}(n, \psi_n) ds, \tag{3.25}$$

$$I_{122}(n, \psi_n) := \gamma \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} J_{n,\psi_n}(x, n-s, s) \bar{I}_{122}(n, \psi_n) ds, \tag{3.26}$$

$$I_{123}(n, \psi_n) := \gamma \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} J_{n,\psi_n}(x, n-s, s) \bar{I}_{123}(n, \psi_n) ds. \tag{3.27}$$

From Li and Xiao [14, (3.45), (3.47) and (3.48)] we derive that

$$I_{121}(n, \psi_n) \leq \frac{2\delta_n}{\gamma(\gamma - \delta_n)} I_{11}(n, \psi_n), \tag{3.28}$$

and that

$$I_{122}(n, \psi_n) \leq \frac{Kn}{F_n^3} \int_0^1 e^{-n\delta_n s} ds \int_{\mathbb{R}^{2d}} \frac{|\widehat{\phi}(z)|(1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}})^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^2} \frac{|\widehat{\phi}(z')| dz dz'}{\sum_{k=1}^d |z'_k|^{\alpha_k}}; \quad (3.29)$$

$$\begin{aligned} I_{123}(n, \psi_n) &\leq \frac{Kn}{F_n^3} \int_0^1 e^{-n\delta_n s} ds \int_{\mathbb{R}^{2d}} \frac{1 - e^{-n \sum_{k=1}^d |z_k + z'_k|^{\alpha_k}}}{\sum_{k=1}^d |z_k + z'_k|^{\alpha_k}} \frac{1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}}}{\sum_{k=1}^d |z'_k|^{\alpha_k}} \\ &\quad \times \frac{(1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}})^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^2} |\widehat{\phi}(z)| |\widehat{\phi}(z')| |\widehat{\phi}(z + z')| dz dz'. \end{aligned} \quad (3.30)$$

Firstly, since $\delta_n \rightarrow 0$, from (3.20) and (3.28) it follows that

$$I_{121}(n, \psi_n) \rightarrow 0. \quad (3.31)$$

Secondly, applying the fact $F_n^2 = n^\kappa \ln n$ to (3.29) we get that

$$I_{122}(n, \psi_n) \leq \frac{K}{n^\kappa \delta_n F_n} \int_{\mathbb{R}^{2d}} \frac{|\widehat{\phi}(z)|(1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}})^2}{\ln n (\sum_{k=1}^d |z_k|^{\alpha_k})^2} \frac{|\widehat{\phi}(z')|}{\sum_{k=1}^d |z'_k|^{\alpha_k}} dz dz'. \quad (3.32)$$

Since $n^\kappa \delta_n \rightarrow \theta \in (0, \infty)$, from (3.17), (3.18), (3.32) and Lemma 2.1 we derive that

$$I_{122}(n, \psi_n) \rightarrow 0. \quad (3.33)$$

At last, using the fact $F_n^2 = n^\kappa \ln n$ again, from (3.30) we get that for some constant $K > 0$,

$$\begin{aligned} I_{123}(n, \psi_n) &\leq \frac{K}{n^\kappa \delta_n n^{\frac{\kappa}{2}} (\ln n)^{3/2}} \int_{\mathbb{R}^{2d}} \frac{1 - e^{-n \sum_{k=1}^d |z_k + z'_k|^{\alpha_k}}}{\sum_{k=1}^d |z_k + z'_k|^{\alpha_k}} \frac{1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}}}{\sum_{k=1}^d |z'_k|^{\alpha_k}} \\ &\quad \times \frac{(1 - e^{-n \sum_{k=1}^d |z_k|^{\alpha_k}})^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^2} |\widehat{\phi}(z)| |\widehat{\phi}(z')| |\widehat{\phi}(z + z')| dz dz'. \end{aligned}$$

Furthermore, by using the inequality $1 - e^{-x} \leq x^{\kappa/8}$ for $x \geq 0$ we have that

$$\begin{aligned} I_{123}(n, \psi_n) &\leq \frac{K}{n^\kappa \delta_n (\ln n)^{3/2}} \int_{\mathbb{R}^{2d}} \frac{|\widehat{\phi}(z + z')|}{(\sum_{k=1}^d |z_k + z'_k|^{\alpha_k})^{1-\kappa/8}} \frac{|\widehat{\phi}(z')|}{(\sum_{k=1}^d |z'_k|^{\alpha_k})^{1-\kappa/8}} \\ &\quad \times \frac{|\widehat{\phi}(z)|}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-\kappa/4}} dz dz'. \end{aligned} \quad (3.34)$$

Since $\bar{\alpha} = 2$, from Lemma 2.1 we know

$$\int_{\mathbb{R}^d} \frac{|\widehat{\phi}(z)|^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-\kappa/4}} dz < \infty,$$

and hence by Hölder inequality,

$$\int_{\mathbb{R}^d} \frac{|\widehat{\phi}(z')| |\widehat{\phi}(z + z')|}{(\sum_{k=1}^d |z_k + z'_k|^{\alpha_k})^{1-\kappa/8} (\sum_{k=1}^d |z'_k|^{\alpha_k})^{1-\kappa/8}} dz'$$

is bounded for all $z \in \mathbb{R}^d$. Therefore

$$\int_{\mathbb{R}^{2d}} \frac{|\widehat{\phi}(z + z')|}{(\sum_{k=1}^d |z_k + z'_k|^{\alpha_k})^{1-\kappa/8}} \frac{|\widehat{\phi}(z')|}{(\sum_{k=1}^d |z'_k|^{\alpha_k})^{1-\kappa/8}} \frac{|\widehat{\phi}(z)|}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-\kappa/4}} dz dz' < \infty.$$

Hence, (3.34) and the fact that $n^\kappa \delta_n \rightarrow \theta \in (0, \infty)$ imply that as $n \rightarrow \infty$,

$$I_{123}(n, \psi_n) \rightarrow 0. \quad (3.35)$$

Consequently, from (3.24)-(3.31), (3.33), (3.35) and (3.14) we have that

$$I_{12}(n, \psi_n) \rightarrow 0. \quad (3.36)$$

Combining (3.20) and (3.36) with (3.12) we derive that as $n \rightarrow \infty$

$$I_1(n, \psi_n) \rightarrow \frac{\gamma\kappa}{\theta(2\pi)^d} \int_{\mathbb{R}^d} \frac{dy}{(1 + \sum_{k=1}^d |y_k|^{\alpha_k})^3} \left(\int_{\mathbb{R}^d} \phi(x) dx \int_0^1 h(t) dt \right)^2. \quad (3.37)$$

Step 2 We are going to get limits of $I_2(n, \psi_n)$ and $I_3(n, \psi_n)$. Let

$$\begin{aligned} \tilde{I}_2(n, \psi_n) &= \int_{\mathbb{R}^d} dx \int_0^n e^{-\delta_n s} \psi_n(x, s) J_{n, \psi_n}(x, n-s, s) ds \\ &= \int_0^n e^{-\delta_n s} ds \int_0^{n-s} f_n(v) dv \int_{\mathbb{R}^d} \psi_n(x, s) T_v \psi_n(x, s+v) dx. \end{aligned} \quad (3.38)$$

From (3.5) and (3.10), it follows that

$$I_2(n, \psi_n) \leq \tilde{I}_2(n, \psi_n). \quad (3.39)$$

Furthermore, by (1.2), (2.4), (2.7) and (3.7), there exists $K > 0$ such that

$$\begin{aligned} I_2(n, \psi_n) &\leq \frac{K}{F_n^2} \int_0^n e^{-\delta_n s} ds \int_0^{n-s} dv \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 e^{-v \sum_{k=1}^d |z_k|^{\alpha_k}} dz \\ &= \frac{K}{F_n^2} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \int_0^1 e^{-n\delta_n s} n ds \int_0^{1-s} e^{-nv \sum_{k=1}^d |z_k|^{\alpha_k}} n dv \\ &\leq \frac{nK}{F_n^2} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz \int_0^1 e^{-n\delta_n s} ds. \end{aligned} \quad (3.40)$$

Since $\bar{\alpha} = 2$ and $F_n^2 = n^\kappa \ln n$, (3.40) and Lemma 2.1 imply that

$$I_2(n, \psi_n) \leq \frac{K}{\ln n} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz \int_0^{n^{1-\kappa}} e^{-n^\kappa \delta_n s} ds \rightarrow 0. \quad (3.41)$$

To get the limit of $I_3(n, \psi_n)$, we let

$$\begin{aligned} \tilde{I}_3(n, \psi_n) &:= \delta_n \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} du \\ &\quad \times \int_{\mathbb{R}^d} \mathbb{E}_x \left(\int_0^u \psi_n(\xi_n(v), s+v) dv \psi_n(\xi_n(u), s+u) \right) dx. \end{aligned} \quad (3.42)$$

Then by (3.6), (3.9) and (3.42), we have that

$$I_3(n, \psi_n) \leq \tilde{I}_3(n, \psi_n), \quad (3.43)$$

and by (2.4), (2.7), (3.7) and (3.42), we have that

$$\begin{aligned} \tilde{I}_3(n, \psi_n) &\leq \frac{\delta_n}{F_n^2 (2\pi)^d} \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} du \int_0^u \tilde{h}\left(\frac{s+v}{n}\right) \tilde{h}\left(\frac{s+u}{n}\right) dv \\ &\quad \times \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 e^{-(u-v) \sum_{k=1}^d |z_k|^{\alpha_k}} dz; \end{aligned} \quad (3.44)$$

see also Li and Xiao [14, (3.68)]. Since \tilde{h} is bounded and $\int_{\mathbb{R}^d} |\widehat{\phi}(z)|^2 dz < \infty$, (3.43) and (3.44) yield that

$$0 \leq I_3(n, \psi_n) \leq K \frac{\delta_n}{F_n^2} \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} u du \leq \frac{K}{\gamma F_n^2},$$

for some constant $K > 0$. Therefore, $F_n \rightarrow \infty$ indicates that

$$I_3(n, \psi_n) \rightarrow 0. \quad (3.45)$$

To get the left hand side of (3.2), we substitute (3.37), (3.41) and (3.45) into (3.3) and obtain that as $n \rightarrow \infty$,

$$\begin{aligned} \mathbb{E}(e^{-\langle \tilde{X}_n, \psi \rangle}) &\rightarrow \exp \left\{ \frac{\gamma \kappa}{\theta (2\pi)^d} \int_{\mathbb{R}^d} \frac{dy}{(1 + \sum_{k=1}^d |y_k|^{\alpha_k})^3} \left(\int_{\mathbb{R}^d} \phi(x) dx \int_0^1 h(t) dt \right)^2 \right\} \\ &= \exp \left\{ \frac{1}{2} \int_0^1 \int_0^1 \text{Cov}(\langle X(s), \psi(\cdot, s) \rangle, \langle X(t), \psi(\cdot, t) \rangle) ds dt \right\}, \end{aligned} \quad (3.46)$$

where X is the limit process in Theorem 2.1. Therefore (3.2) holds and (i) is proved.

Now we are in the place to prove the tightness of $\{\langle X_n, \phi \rangle, n \geq 1\}$ in $C([\varepsilon, 1], \mathbb{R})$. Note that by the same argument as those used in Bojdecki *et al* [5], we also have that X_n converges to X in finite-dimensional distributions. This implies the tightness of $\{\langle X_n(\varepsilon), \phi \rangle\}$. According to the proof of Proposition 3.3 in [4], the remainder is to prove that for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\varepsilon \leq t_1 < t_2 \leq 1$ and $\eta > 0$, there exist constants $a \geq 1$, $b > 0$ and $K > 0$, which is independent of t_1, t_2 , such that for all $n \geq 1$.

$$\int_0^{1/\eta} \left(1 - \text{Re} \left(\mathbb{E} \left(\exp \{ -i\omega \langle \tilde{X}_n, \phi h \rangle \} \right) \right) \right) d\omega \leq \frac{K}{\eta^a} (t_2 - t_1)^{1+b}, \quad (3.47)$$

where $h \in \mathcal{S}(R)$ is an approximation of $\mathbf{1}_{\{t_2\}}(t) - \mathbf{1}_{\{t_1\}}(t)$ supported on $[t_1, t_2]$ such that $\tilde{h}(t)$ satisfies

$$\tilde{h} \in \mathcal{S}(R), \quad 0 \leq \tilde{h} \leq \mathbf{1}_{[t_1, t_2]}. \quad (3.48)$$

Repeating the discussion on $\mathbb{E}(\exp\{-\langle \tilde{X}_n, \psi \rangle\})$ (see Li and Xiao [14, Section 3]) with ψ replaced by $i\omega\phi h$, we can readily get that

$$\mathbb{E}(\exp\{-i\omega \langle \tilde{X}_n, \phi h \rangle\}) = \exp \{ I_1(n, i\omega\psi_n) + I_2(n, i\omega\psi_n) + I_3(n, i\omega\psi_n) \},$$

and the inequality

$$|V_{n, i\omega\psi_n}| \leq J_{n, \omega\psi_n} = \omega J_{n, \psi_n}.$$

Consequently, from the expressions of I_1, I_2, I_3 and $I_{11}, \tilde{I}_2, \tilde{I}_3$ (see (3.4)-(3.6), (3.13), (3.38) and (3.42), respectively), it is easy to check that the following inequalities hold.

$$\begin{cases} |I_1(n, i\omega\psi_n)| \leq I_{11}(n, \omega\psi_n) = \omega^2 I_{11}(n, \psi_n); \\ |I_2(n, i\omega\psi_n)| \leq \tilde{I}_2(n, \omega\psi_n) = \omega^2 \tilde{I}_2(n, \psi_n); \\ |I_3(n, i\omega\psi_n)| \leq \tilde{I}_3(n, \omega\psi_n) = \omega^2 \tilde{I}_3(n, \psi_n). \end{cases} \quad (3.49)$$

(I) We first estimate the upper bound of $I_{11}(n, \psi_n)$. Substituting (3.7), (3.10) and (1.2) into (3.13), we get that for some constant $K > 0$

$$\begin{aligned} I_{11}(n, \psi_n) &\leq \frac{K}{F_n^2} \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\delta_n u} \tilde{h}\left(\frac{s+u}{n}\right) du \\ &\quad \times \int_0^{n-s} e^{-\delta_n v} \tilde{h}\left(\frac{s+v}{n}\right) dv \int_{\mathbb{R}^d} T_u \phi(x) T_v \phi(x) dx. \end{aligned}$$

Furthermore, by using (2.4) and (2.7), we have that

$$\begin{aligned} I_{11}(n, \psi_n) &\leq \frac{2Kn^3}{F_n^2} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \int_0^1 \tilde{h}(u) du \int_0^u \tilde{h}(v) e^{-n\delta_n(u+v)} dv \\ &\quad \times \int_0^v e^{n\delta_n s} e^{-n(u+v-2s) \sum_{k=1}^d |z_k|^{\alpha_k}} ds \end{aligned}$$

which and the condition (3.48) imply that $I_{11}(n, \psi_n)$ is bounded from above by

$$\begin{aligned} &\frac{2Kn^3}{F_n^2} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \int_{t_1}^{t_2} du \int_{t_1}^u e^{-n(u+v)(\delta_n + \sum_{k=1}^d |z_k|^{\alpha_k})} dv \\ &\quad \times \int_0^v e^{ns(2 \sum_{k=1}^d |z_k|^{\alpha_k} + \delta_n)} ds \end{aligned}$$

which is further bounded from above by

$$\begin{aligned} &\frac{2Kn^2}{F_n^2} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{2 \sum_{k=1}^d |z_k|^{\alpha_k} + \delta_n} dz \int_{t_1}^{t_2} e^{-n\delta_n u} du \int_{t_1}^u e^{-n(u-v) \sum_{k=1}^d |z_k|^{\alpha_k}} dv \\ &\leq \frac{Kn}{F_n^2} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz \int_{t_1}^{t_2} e^{-n\delta_n u} \frac{1 - e^{-n(u-t_1) \sum_{k=1}^d |z_k|^{\alpha_k}}}{\sum_{k=1}^d |z_k|^{\alpha_k}} du. \end{aligned} \quad (3.50)$$

Since $t_2 \geq t_1 \geq \varepsilon$, using the inequality $1 - e^{-x} \leq x^r$ for all $x \geq 0$ and $r \in (0, 1]$, from (3.50) we get that

$$I_{11}(n, \psi_n) \leq \frac{Kn^{1+r}}{F_n^2} e^{-n\delta_n \varepsilon} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-r}} dz \int_{t_1}^{t_2} (u - t_1)^r du,$$

for every $r \in (0, 1)$. Lemma 2.1 implies $\int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{(\sum_{k=1}^d |z_k|^{\alpha_k})^{2-r}} dz < \infty$. In addition, $n^\kappa \delta_n \rightarrow \theta \in (0, \infty)$ with $\kappa \in (0, 1)$ implies that for all n , $\frac{Kn^{1+r}}{F_n^2} e^{-n\delta_n \varepsilon}$ are bounded. Consequently, there exists a constant K independent of t_1 and t_2 , such that

$$I_{11}(n, \psi_n) \leq K |t_2 - t_1|^{1+r}. \quad (3.51)$$

(II) We then proceed to estimate $\tilde{I}_2(n, \psi_n)$. Since \bar{f}_n is bounded, applying (3.7), (3.10) and (1.2) to (3.38) we obtain that for some constant $K > 0$,

$$\tilde{I}_2(n, \psi_n) \leq \frac{K}{F_n^2} \int_0^n e^{-\delta_n s} \tilde{h}\left(\frac{s}{n}\right) ds \int_0^{n-s} e^{-\delta_n u} \tilde{h}\left(\frac{s+u}{n}\right) du \int_{\mathbb{R}^d} \phi(x) T_u \phi(x) dx.$$

Then by the same arguments as led to (3.50), we have that

$$\begin{aligned} \tilde{I}_2(n, \psi_n) &\leq \frac{Kn^2}{F_n^2} \int_{t_1}^{t_2} e^{-n\delta_n s} ds \int_s^{t_2} e^{-n\delta_n(u-s)} du \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 e^{-n(u-s) \sum_{k=1}^d |z_k|^{\alpha_k}} dz \\ &\leq \frac{Kn}{F_n^2} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{\sum_{k=1}^d |z_k|^{\alpha_k}} dz \int_{t_1}^{t_2} e^{-n\delta_n s} (1 - e^{-n(t_2-s) \sum_{k=1}^d |z_k|^{\alpha_k}}) ds. \end{aligned} \quad (3.52)$$

Consequently, repeating the same arguments used to (3.51), we can readily get that for any $r \in (0, 1)$, there exists a constant $K > 0$ independent of t_1 and t_2 such that

$$\tilde{I}_2(n, \psi_n) \leq K|t_2 - t_1|^{1+r}. \quad (3.53)$$

(III) At last, we consider $\tilde{I}_3(n, \psi_n)$. Since $\delta_n \rightarrow 0$, without loss of generality, we can assume $\delta_n < \gamma$. Using again (1.2), (3.7), (3.10), (2.4) and (2.7) to (3.44), from the condition (3.48), we get that for any $r \in (0, 1)$,

$$\begin{aligned} \tilde{I}_3(n, \psi_n) &\leq \frac{\delta_n}{F_n^2} \int_0^n e^{-\delta_n s} ds \int_0^{n-s} e^{-\gamma u} du \int_0^u \tilde{h}\left(\frac{s+v}{n}\right) \tilde{h}\left(\frac{s+u}{n}\right) dv \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \\ &= \frac{n^3 \delta_n}{F_n^2} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \int_0^1 e^{-n\delta_n s} ds \int_s^1 e^{-n\gamma(u-s)} \tilde{h}(u) du \int_s^u \tilde{h}(v) dv \\ &= \frac{n^3 \delta_n}{F_n^2} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \int_{t_1}^{t_2} e^{-n\gamma u} du \int_{t_1}^u dv \int_0^v e^{ns(\gamma-\delta_n)} ds \\ &\leq \frac{n\delta_n}{(\gamma-\delta_n)^2 F_n^2} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \int_{t_1}^{t_2} e^{-n\delta_n t_1} (1 - e^{-n\gamma(u-t_1)}) du \\ &\leq \frac{\gamma^r n^{1+r} \delta_n}{(\gamma-\delta_n)^2 F_n^2} e^{-n\delta_n \varepsilon} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz |t_2 - t_1|^{1+r}. \end{aligned} \quad (3.54)$$

By the same reason as applied to (3.51), for every $r \in (0, 1)$ there exists a constant K independent of t_1 and t_2 such that

$$\tilde{I}_3(n, \psi_n) \leq K|t_2 - t_1|^{1+r}. \quad (3.55)$$

Summing up, from (3.51), (3.53) and (3.55) we know that for any $r \in (0, 1)$, there is a constant K which is independent of t_1, t_2 such that

$$|\tilde{I}_3(n, i\omega\psi_n)| + |\tilde{I}_2(n, i\omega\psi_n)| + |I_{11}(n, i\omega\psi_n)| \leq K(\phi, r)\omega^2|t_2 - t_1|^{1+r}. \quad (3.56)$$

Note that

$$\begin{aligned} &\left| 1 - \operatorname{Re} \left(\mathbb{E} \left(\exp \{ -i\omega \langle \tilde{X}_n, \phi h \rangle \} \right) \right) \right| \\ &\leq |I_1(n, i\omega\psi_n)| + |I_2(n, i\omega\psi_n)| + |I_3(n, i\omega\psi_n)|. \end{aligned} \quad (3.57)$$

Therefore, (3.49), (3.56) and (3.57) yield that

$$\int_0^{1/\eta} \left(1 - \operatorname{Re} \left(\mathbb{E} \left(\exp \{ -i\omega \langle \tilde{X}_n, \phi h \rangle \} \right) \right) \right) d\omega \leq \frac{K(\phi, r)}{3\eta^3} |t_2 - t_1|^{1+r},$$

which completes the proof of (3.47) and hence the proof of Theorem 2.1. \square

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